ALGEBRA-II
Mid-Semestral Exam
2017-2018

Total marks: 100
Time: 3hours

Answer all questions.

(1) State true or false. Justify your answers.
(i) Suppose that A and B are nonsingular $n \times n$ matrices. Then $A+B$ is nonsingular.
(ii) Let A be an $n \times n$ matrix such that the sum of elements in each row of A is zero. Then A is a singular matrix.
(iii) Let $C[\pi, \pi]$ be the vector space of all continuous functions defined on the interval $[\pi, \pi]$. The subset $\{cos(x), sin(x)\}$ in $\mathbb{C}[\pi, \pi]$ is linearly independent.
(iv) Let A be an $n \times n$ matrix. If rank$(A) \neq n$, then 0 is an eigenvalue of A.
(v) Let A, B be $n \times n$ matrices with A invertible. Then nullity(AB) = nullity(B).
. [5× 5]

(2) Let

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 1 & 2 & 5 & 4 \end{bmatrix}.$$

(a) Find the solutions to the system of equations $AX = 0$.
(b) Find a basis for the solution space of $AX = 0$. Justify your answer. [6+4]

(3) Let P_3 denote the vector space of all polynomials with real coefficients of degree less than or equal to 3. Let $S = \{1+x, 1+x^2, x-x^2+2x^3, 1-x-x^2\}$.
(a) Show that S is a basis for P_3.
(b) Find the change of basis matrix with respect to the standard basis $\{1, x, x^2, x^3\}$.
(c) Find the co-ordinate vector for the polynomial $f(x) = -3 + 2x^3$ in terms of the basis S. [8+6+6]

(4) The *trace* of a square matrix is the sum of its diagonal entries. Let W_1 be the space of $n \times n$ matrices with trace zero. Find a subspace W_2 of $\mathbb{R}^{n \times n}$ such that $\mathbb{R}^{n \times n} = W_1 \oplus W_2$. [10]

(5) (a) State and prove *Dimension Formula* (rank-nullity theorem) for a linear transformation $T : V \longrightarrow W$, where V is a finite dimensional vector space.
(b) Let $T : V \longrightarrow V$ be a linear operator on a vector space of dimension 2. Assume that T is not multiplication by a scalar. Prove that there is a vector v in V such that $\mathcal{B} = (v, T(v))$ is a basis of V, and describe the matrix of T with respect to that basis. [10+5]

(6) Define the map $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_2 + x_3 \\ x_3 \end{bmatrix}$$

(a) Find the matrix of T with respect to the standard basis.
(b) Find the matrix of T with respect to the basis $\mathcal{B} = \{(1, 1, 0)^t, (1, -1, 0)^t, (-1, 0, -1)^t\}$.
(c) Describe the null space (kernel) and the range of T. Also find the rank and the nullity of T. [4+8+8]

—————- End —————